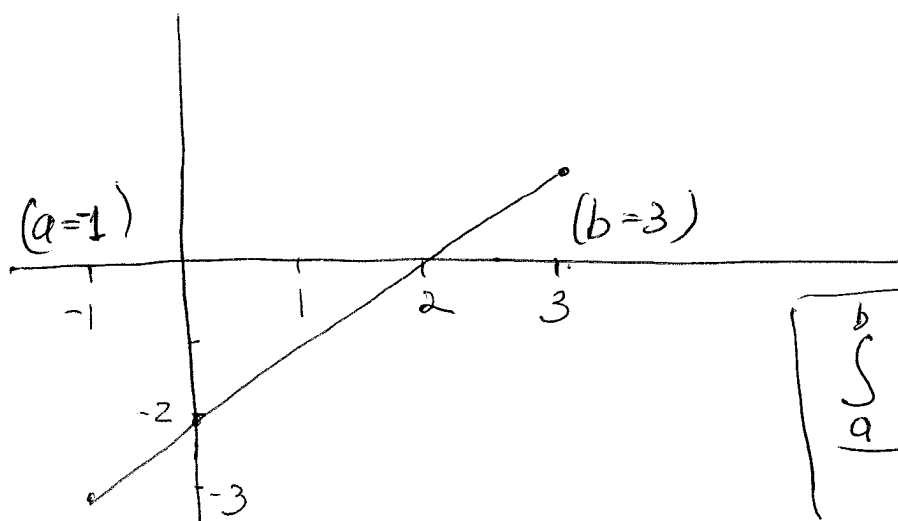


Example: $f(x) = x - 2$, $x \in [-1, 3] = [a, b]$

Find \bar{f} .



$$\boxed{\frac{\int_a^b f(x) dx}{b-a} = \bar{f}}$$

$$\bar{f} = \frac{\int_{-1}^3 (x-2) dx}{3 - (-1)} = \frac{\left. \frac{x^2}{2} \right|_{-1}^3 - 2x \Big|_{-1}^3}{4} =$$

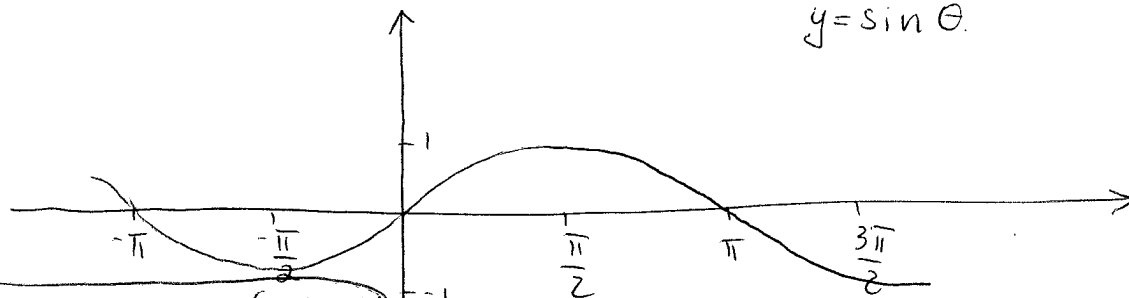
$$= \frac{\frac{9}{2} - \frac{(-1)^2}{2} - 2(3 - (-1))}{4} = \frac{4 - 2 \cdot 4}{4} = -1$$

$$\left(\int_{-1}^3 f(x) dx = \int_{-1}^3 (x-2) dx = -4 \right)$$

$$\int_{-1}^3 \bar{f} dx = \int_{-1}^3 (-1) dx =$$

$$\int_{-1}^3 dx = x \Big|_{-1}^3 = (-1) - 3 = -4 = \bar{f} \cdot (b-a) = -1 \cdot 4 = -4$$

$$y = \sin \theta$$



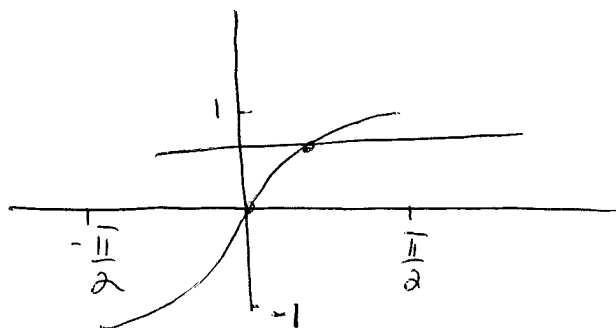
a function: (input) \rightarrow (output)

$$\sin : \begin{array}{ccc} \theta & \rightarrow & y \\ \uparrow & & \uparrow \\ \text{an angle} & & \text{a number} \end{array}$$

$y = \sin \theta$, the domain is $\mathbb{R} : -\infty < \theta < \infty$
 $\theta \in \mathbb{R}$

the range is $[-1, 1]$, $y \in [-1, 1]$

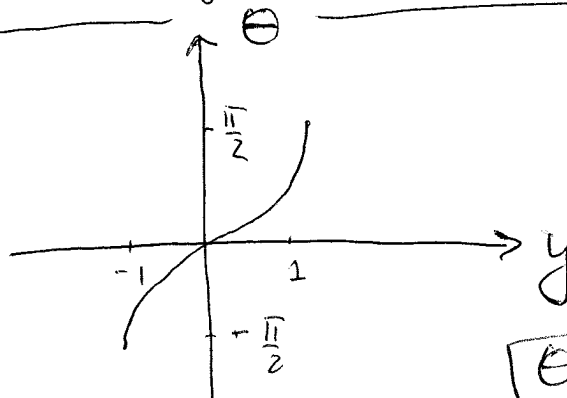
However, $y = \sin \theta$ is not one-to-one, because the horizontal line test is not satisfied.
 Let's restrict the domain. Consider $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.



Now, the horizontal test is satisfied and the restricted function is invertible.

$$\sin^{-1} \text{ or arcsin} : \begin{array}{ccc} y & \rightarrow & \theta \\ \text{a number} & & \text{an angle} \end{array}$$

$\theta = \arcsin y$, the domain is $[-1, 1]$
 the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$



(We reflected the graph of $\sin \theta$ about the diagonal)

$$\theta = \arcsin y$$

Goal:

$$(\arcsin y)' = ?$$

$\sin(\arcsin y) = y$ \leftarrow differentiate both sides with respect to y .

(On the LHS, we use the chain rule)

$$(\sin(\arcsin y))' = 1$$

$$\cos(\arcsin y) \cdot (\arcsin y)' = 1$$

$$(\arcsin y)' = \frac{1}{\cos(\arcsin y)} = \frac{1}{\cos(\theta)}$$

Remember: $\cos^2 \theta + \sin^2 \theta = 1$

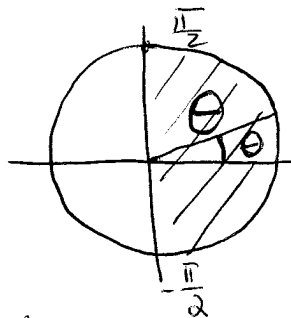
$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

Since we have the restriction on θ :

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos \theta \geq 0.$$



Thus, we exclude

$$\cos \theta = -\sqrt{1 - \sin^2 \theta} \quad (\text{which is a negative number})$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{Going back to } (\arcsin y)' = \frac{1}{\cos \theta} = \frac{1}{\sqrt{1 - \sin^2 \theta}} =$$

$$= \frac{1}{\sqrt{1 - \sin^2(\arcsin y)}} = \frac{1}{\sqrt{1 - (\sin(\arcsin y))^2}} = \frac{1}{\sqrt{1 - y^2}}$$

$$(\arcsin y)' = \frac{1}{\sqrt{1 - y^2}}$$

\Rightarrow

$$\int \frac{dy}{\sqrt{1 - y^2}} = \arcsin(y) + C$$

Remark 2

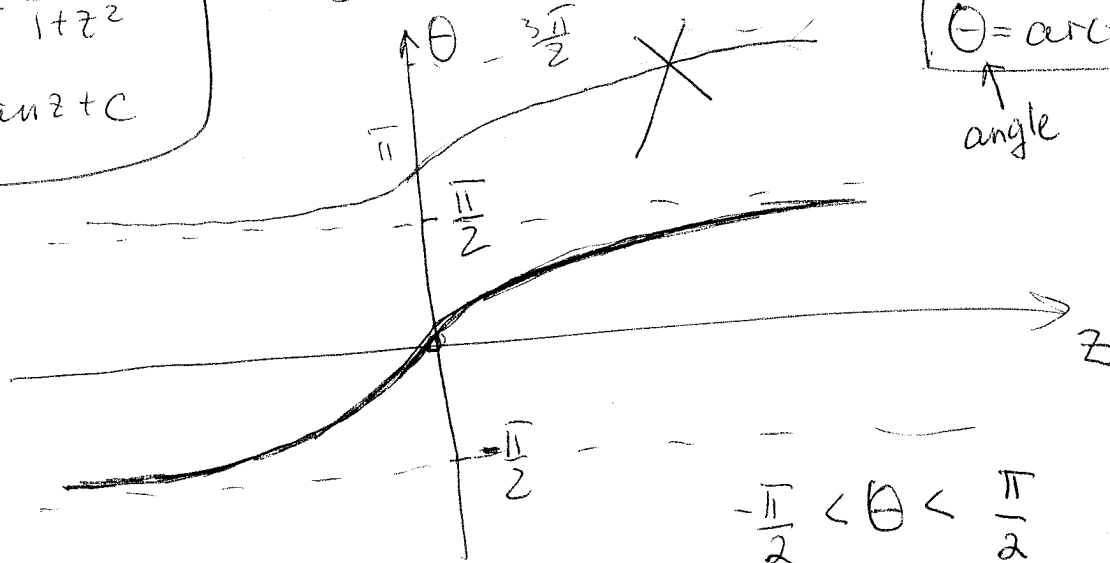
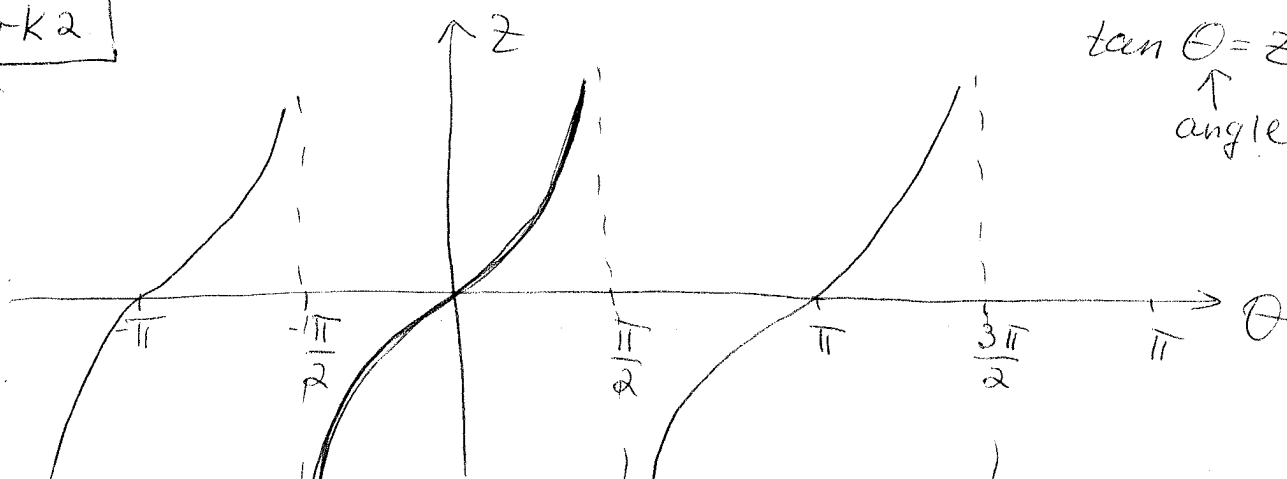
Want to show that
 $(\arctan z)' = \frac{1}{1+z^2}$ or
 $\int \frac{dz}{1+z^2} = \arctan z + C$

reflect curves about

$z = \theta$:

$\boxed{\theta = \arctan z}$
 ↑ angle ↑ number

$\tan \theta = z$
 ↑ angle ↑ number

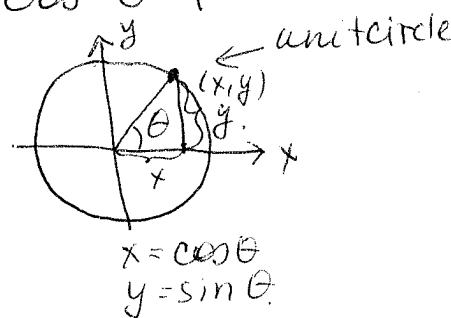


Preliminaries:

(a) we use the following equality $\sin^2 \theta + \cos^2 \theta = 1$

multiply by $\frac{1}{\cos^2 \theta}$:

$$\tan^2 \theta + 1 = \frac{1}{\cos^2 \theta} \quad \text{or} \quad \cos^2 \theta = \frac{1}{1 + \tan^2 \theta} \quad \textcircled{A}$$



$$(b) (\tan \theta)' = \left(\frac{\sin \theta}{\cos \theta} \right)' = \frac{1}{\cos^2 \theta}$$

To calculate $(\arctan z)'$, we differentiate $\tan(\arctan z) = z$ with respect to z :

$$\frac{1}{\cos^2(\arctan z)} \cdot (\arctan z)' = 1 \quad \text{or} \quad \textcircled{A}$$

$$(\arctan z)' = \cos^2(\arctan z)$$

$$= \frac{1}{1 + (\tan(\arctan z))^2} = \frac{1}{1 + z^2} \Rightarrow \boxed{\int \frac{dz}{1+z^2} = \arctan z + C}$$

Evaluate $\int_0^1 \arctan x \, dx$

Use integration by parts:

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

$u = \arctan x$ $\frac{du}{dx} = \frac{1}{1+x^2}$ $du = \frac{dx}{1+x^2}$	$dx = dv$ $x = v$
---	----------------------

$$\int_0^1 \arctan x \, dx = x \arctan x \Big|_0^1 - \int_0^1 \frac{x \, dx}{1+x^2} =$$

$$= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_0^1 \frac{x \, dx}{1+x^2}$$

\Rightarrow For $\int_0^1 \frac{x \, dx}{1+x^2}$ = use substitution

$1+x^2 = y(x)$ $2x \, dx = dy$ $x \, dx = \frac{dy}{2}$	<table border="1"> <tr> <th>x</th> <th>0</th> <th>1</th> </tr> <tr> <th>y</th> <td>1</td> <td>2</td> </tr> </table>	x	0	1	y	1	2
x	0	1					
y	1	2					

$$= \int_1^2 \frac{dy}{2 \cdot y} = \frac{1}{2} \int_1^2 \frac{dy}{y} = \frac{1}{2} \ln|y| \Big|_1^2 = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

back to the original integral:

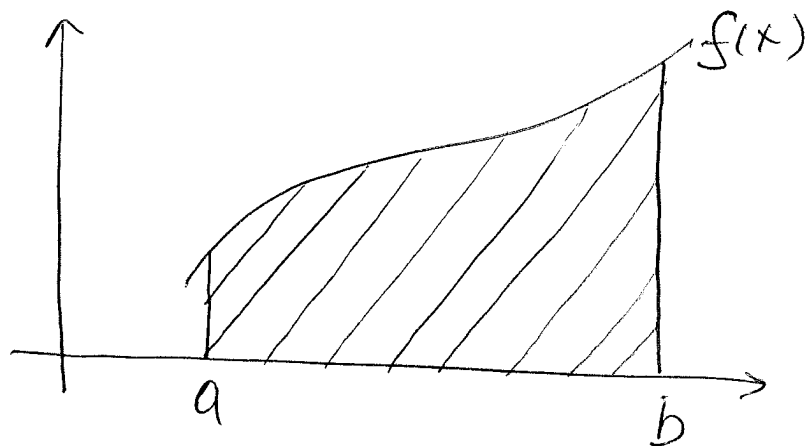
$$\int_0^1 \arctan x \, dx = 1 \cdot \frac{\pi}{4} - 0 - \frac{1}{2} \ln 2 = \frac{\pi}{4} - \frac{1}{2} \ln 2 \approx 0.439$$

$\int_0^1 \frac{e^{\arctan x}}{1+x^2} =$ substitution	$\arctan x = u(x)$ $\frac{dx}{1+x^2} = du$	<table border="1"> <tr> <th>x</th> <th>0</th> <th>1</th> </tr> <tr> <th>u</th> <td>0</td> <td>$\frac{\pi}{4}$</td> </tr> </table>	x	0	1	u	0	$\frac{\pi}{4}$
x	0	1						
u	0	$\frac{\pi}{4}$						

$$= \int_0^{\frac{\pi}{4}} e^u \, du = e^u \Big|_0^{\frac{\pi}{4}} = e^{\frac{\pi}{4}} - e^0 = e^{\frac{\pi}{4}} - 1$$

Integrals and Areas

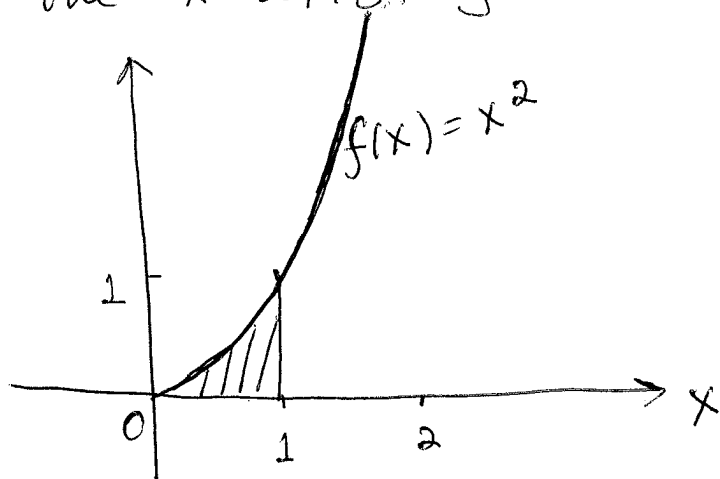
From Lecture 1, we know that if $f(x)$ is a continuous, nonnegative function ($f(x) \geq 0$) on $[a, b]$ then $\int_a^b f(x) dx =$ area between the x -axis and the graph of $f(x)$ between $x=a$ and $x=b$.



(follows from the Riemann sums)

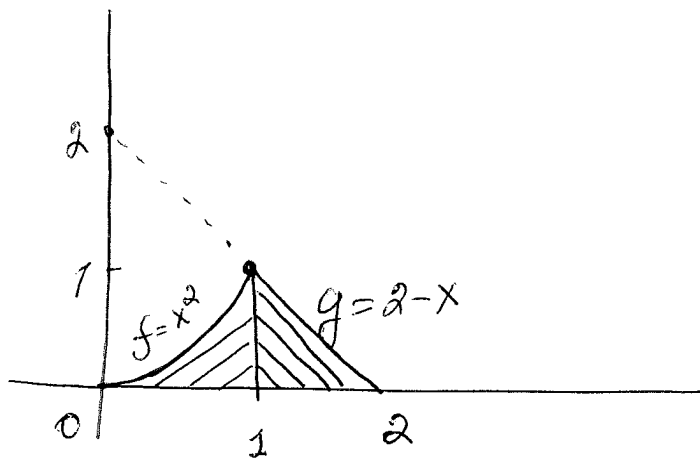
Example 1: $y = x^2 = f(x)$

Find the area between the graph of $f(x)$, the x -axis, from $x=0$ to $x=1$.



$$\text{Area} = \int_0^1 x^2 dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

Let's make the problem harder:
consider a piecewise, nonnegative function



This is a continuous function for $0 \leq x \leq 2$.

First, we need to find the point of intersection where the two curves intersect; we solve

$$x^2 = 2 - x$$

$$x^2 + x - 2 = 0$$

$$\Delta = 1 - 4(-2) = 9$$

$$x_1 = \frac{-1+3}{2} = 1$$

$$x_2 = \frac{-1-3}{2} = -2 \leftarrow$$

$x_2 = -2$ does not belong to the interval of integration:

$$[0, 2]$$

We consider the shaded region as a sum of the two regions: between $f(x)$ and the x-axis, $0 \leq x \leq 1$ and between $g(x)$, the x-axis, $1 \leq x \leq 2$.

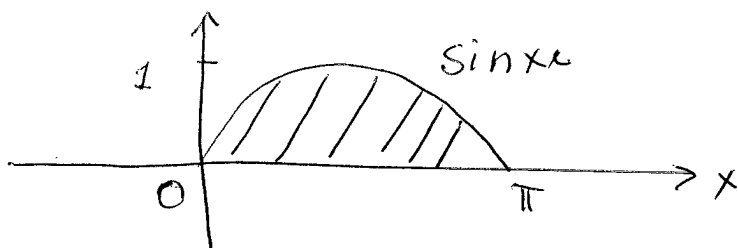
$$\begin{aligned} \text{Area} &= \text{Area 1} + \text{Area 2} = \int_0^1 x^2 dx + \int_1^2 (2-x) dx = \\ &= \left. \frac{x^3}{3} \right|_0^1 + \left. \left(2x - \frac{x^2}{2} \right) \right|_1^2 = \frac{1}{3} + \left(2 \cdot 2 - \frac{2^2}{2} \right) - \left(2 \cdot 1 - \frac{1^2}{2} \right) = \end{aligned}$$

$$= \frac{1}{3} + (4-2) - (2-0.5) = \frac{1}{3} + 2 - \frac{3}{2} =$$

$$= \frac{2+12-9}{6} = \frac{5}{6}$$

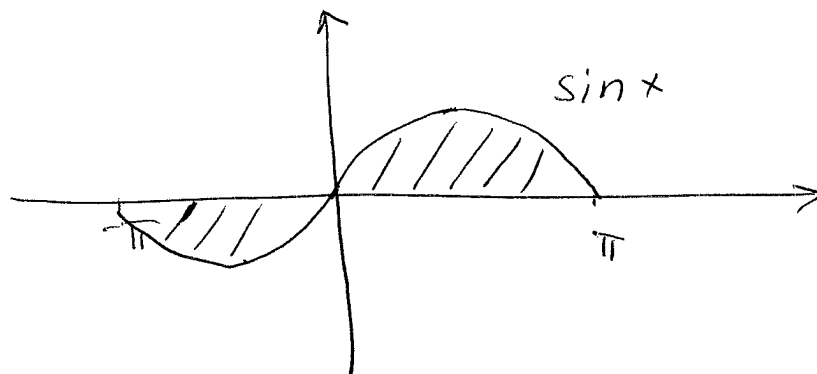
$$\int_0^{\pi} \sin x dx = -\cos x \Big|_0^{\pi} = -\cos \pi - (-\cos 0) =$$

$$-(-1) + 1 = 2$$



Consider $-\pi \leq x \leq \pi$

$$\int_{-\pi}^{\pi} \sin x dx = \int_{-\pi}^0 \sin x dx + \int_0^{\pi} \sin x dx =$$



the previous example
↓

$$= -\cos x \Big|_{-\pi}^0 - \cos x \Big|_0^{\pi} = -\cos(0) + \cos(-\pi) + 2 =$$

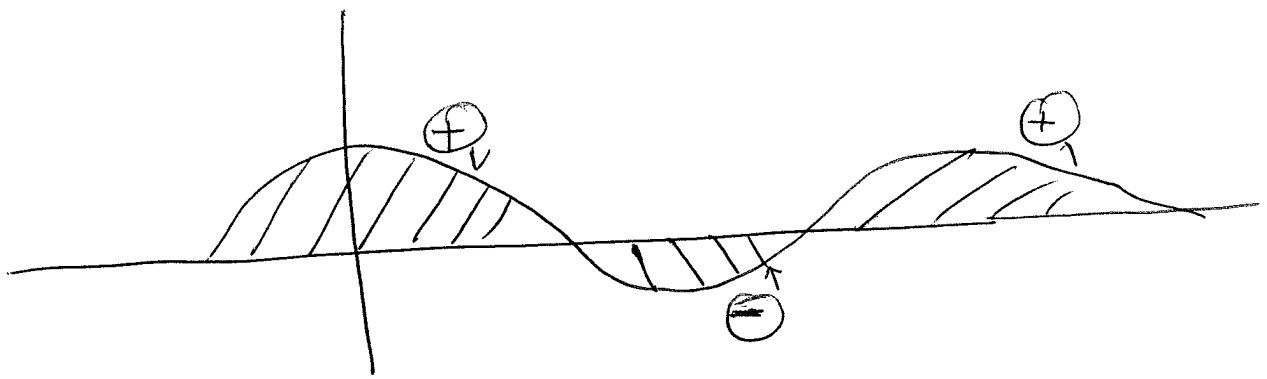
$$= -1 - 1 + 2 = 0$$

How can an area be ~~neg~~ zero?

From Lecture 1, we know that when $f(x)$ is positive for some values of x and negative for others and $a < b$ then

$$\int_a^b f(x) dx = \left[\begin{array}{l} \text{sum of the areas of the regions} \\ \text{above the x-axis} \end{array} \right]$$

$$- \left[\begin{array}{l} \text{sum of the areas of the regions below} \\ \text{the x-axis} \end{array} \right]$$



or

$$\int_a^b f(x) dx = \text{"area above"} - \text{"area below"}$$

$$\int_{-\pi}^{\pi} \sin x dx = \underbrace{\int_{-\pi}^0 \sin x dx}_{\text{negative area} = -2} + \underbrace{\int_0^{\pi} \sin x dx}_{\text{positive area} = 2} = 0.$$

negative area = -2
 ↑
 below the x-axis

↑
 above the x-axis

Thus, the definite integral is the signed area

To find actual area between the graph of $f(x) = \sin x$, the x-axis and $x = -\pi$, $x = \pi$, we consider the following integral:

$$\int_{-\pi}^{\pi} |\sin x| dx = \int_{-\pi}^0 |\sin x| dx + \int_0^{\pi} |\sin x| dx =$$

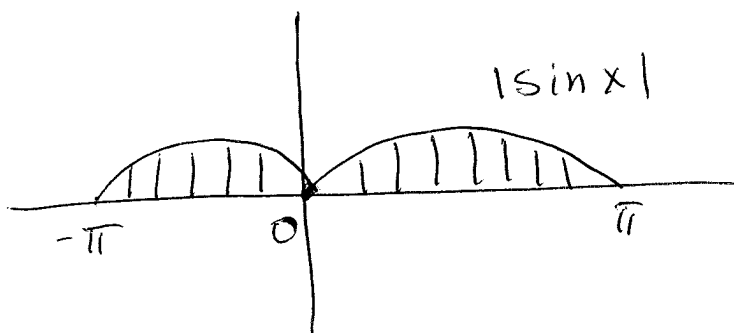
$$= \int_{-\pi}^0 -\sin x dx + \int_0^{\pi} \sin x dx =$$

$$= - \int_{-\pi}^0 \sin x dx + \int_0^{\pi} \sin x dx =$$

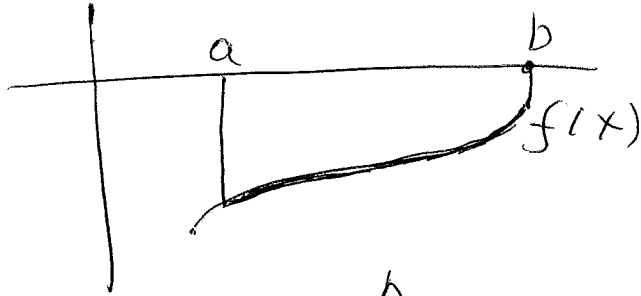
$$\cos x \Big|_{-\pi}^0 - \cos x \Big|_0^{\pi} =$$

$$= \cos 0 - \cos(-\pi) - \cos \pi + \cos 0 =$$

$$= 1 + 1 + 1 + 1 = 4$$



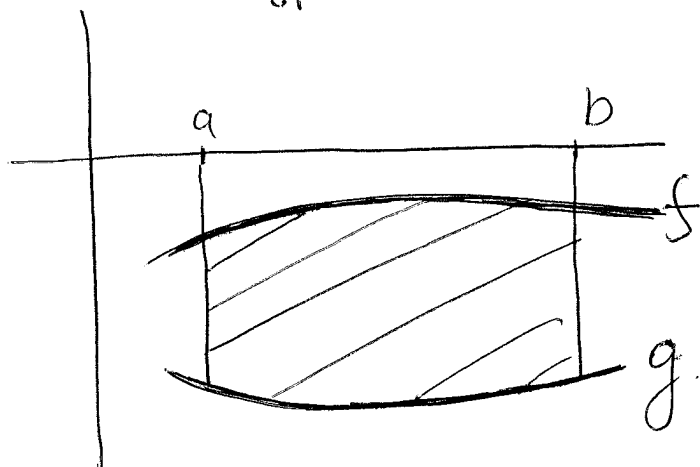
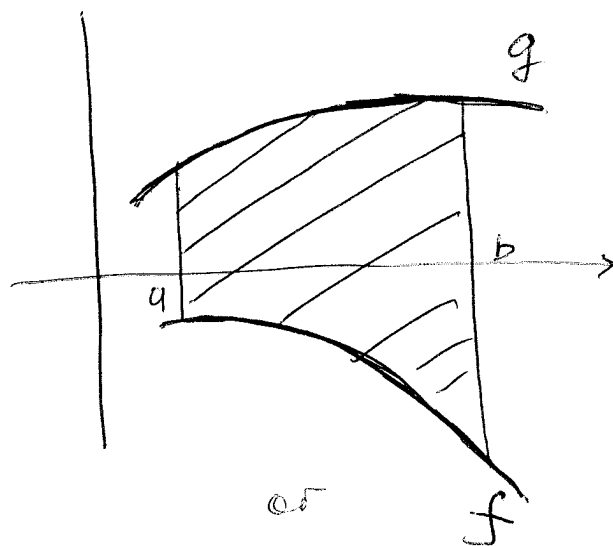
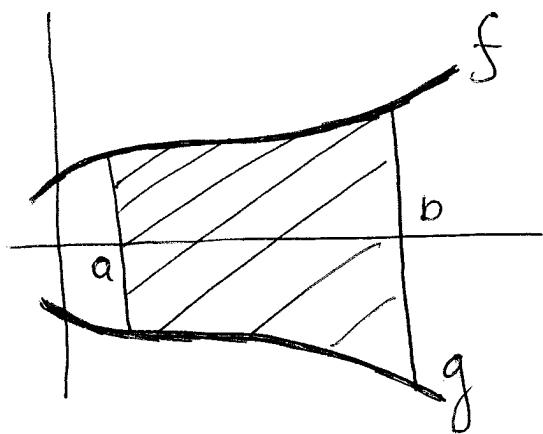
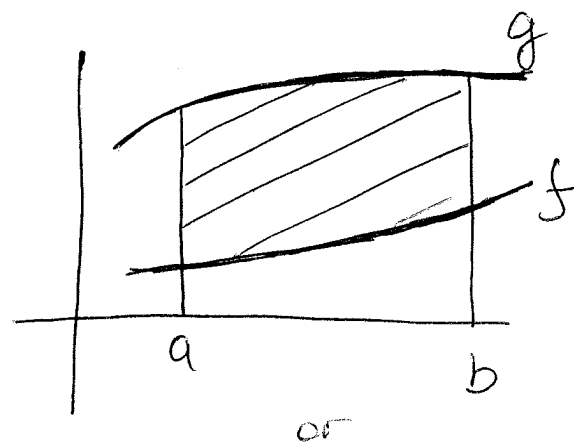
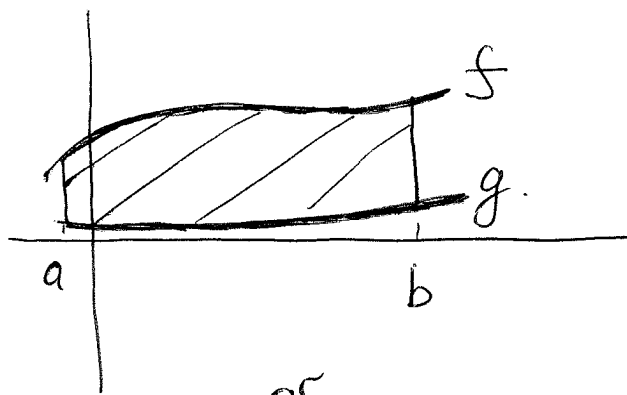
In the general case, if $f(x) \leq 0$.



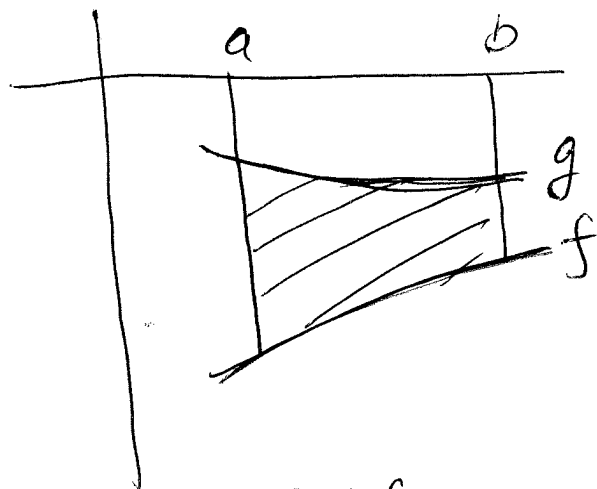
$$\text{Area} = \int_a^b |f(x)| dx = \int_a^b -f(x) dx = - \int_a^b f(x) dx$$

negative
positive

Let's make the problem harder and consider the area of the region between the graphs of continuous functions $f(x)$ and $g(x)$ bounded by the vertical lines $x=a$ and $x=b$.



$$f > g$$



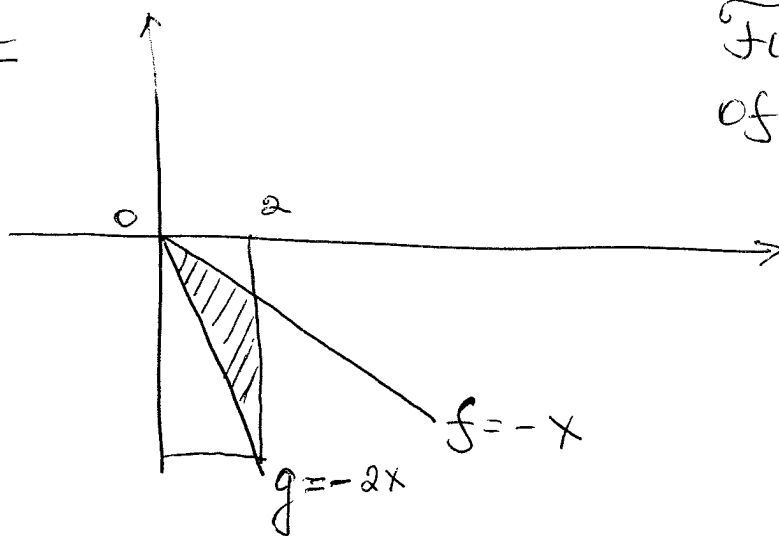
$$g < f$$

Then the area = $\int_a^b |f(x) - g(x)| dx$.

(no point of intersection between f and g , $x \in [a, b]$)

$$\int_a^b |f(x) - g(x)| dx = \begin{cases} \int_a^b (f(x) - g(x)) dx, & \text{if } f(x) > g(x) \text{ for } x \in [a, b] \\ \text{and} \\ \int_a^b (g(x) - f(x)) dx, & \text{if } g(x) > f(x) \text{ for } x \in [a, b] \end{cases}$$

Example



Find the area of the shaded region b/w $-x, -2x$, $0 \leq x \leq 2$

$$\int_0^2 |f(x) - g(x)| dx = \text{take a test point } 0 < \bar{c} < 2$$

$\bar{c} = 1$

$f(1) = -1, \quad g(1) = -2$

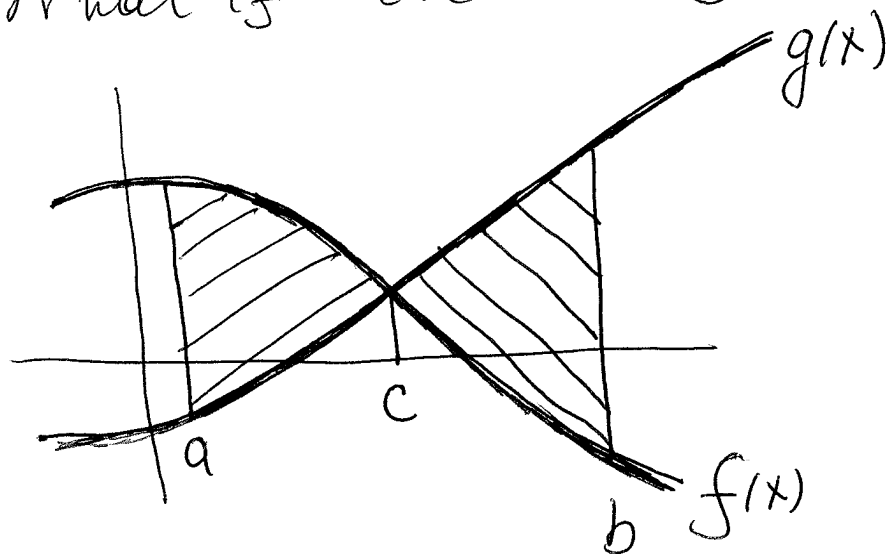
$f(1) > g(1)$

and $f(x) > g(x)$ for $x \in (0, 2)$,

thus $|f(x) - g(x)| = f(x) - g(x)$.

$$\begin{aligned}\int_0^2 |f(x) - g(x)| dx &= \int_0^2 (f(x) - g(x)) dx = \\&= \int_0^2 (-x - (-2x)) dx = \int_0^2 (-x + 2x) dx = \int_0^2 x dx = \\&= \left. \frac{x^2}{2} \right|_0^2 = \frac{2^2}{2} - 0 = 2 \text{ (units}^2\text{)}.\end{aligned}$$

→ What if the two functions intersect?

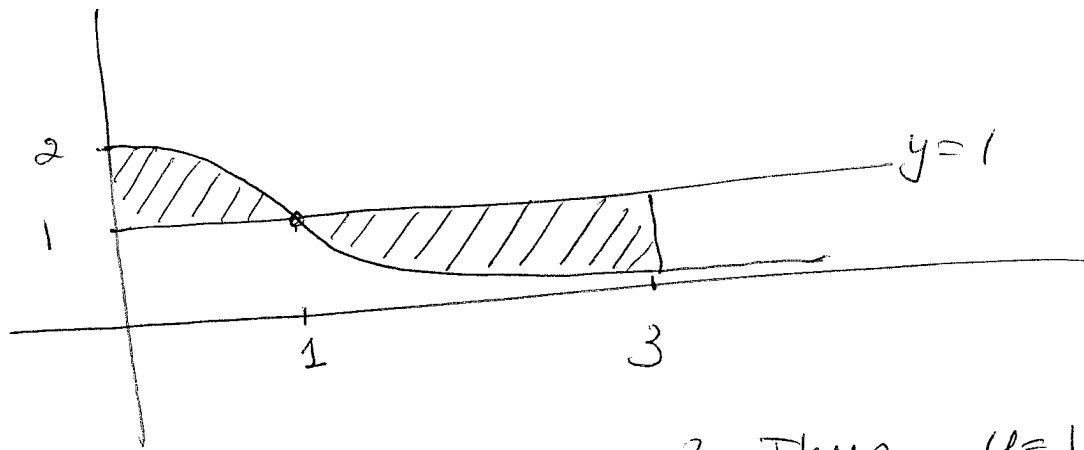


c is the point of intersection.
(common point for both $f(x)$ and $g(x)$:
 $f(c) = g(c)$)

$$\begin{aligned}\int_a^b |f(x) - g(x)| dx &= \int_a^c |f(x) - g(x)| dx + \int_c^b |f(x) - g(x)| dx = \\&= \int_a^c (f(x) - g(x)) dx + \int_c^b (g(x) - f(x)) dx.\end{aligned}$$

Example

Find the area of the region between $y = \frac{2}{1+x^2} = f$ and $y = 1 = g$ and $x=0, x=3$



$$\left\{ \begin{array}{l} y(0) = 2 \\ y(x) = \frac{2}{1+x^2} \rightarrow 0 \text{ as } x \rightarrow \infty \\ y'(x) = \frac{-4x}{(1+x^2)^2} < 0 \text{ for } x > 0 \end{array} \right\} \text{ Thus, } y=1 \text{ intersects } y = \frac{2}{1+x^2} \text{ is decreasing for } x > 0.$$

Points of intersection:

$$\frac{2}{1+x^2} = 1$$

$$2 = 1+x^2$$

$x^2 = 1 \Rightarrow x = \pm 1$ Within the interval $[0, 3]$ the two curves intersect at $x=1$ only.

$$\int_0^3 |f(x) - g(x)| dx = \int_0^1 |f(x) - g(x)| dx + \int_1^3 |f(x) - g(x)| dx$$

$$\ln[0, 1] \quad \bar{c} = 0.5 \quad f(0.5) = \frac{2}{1+0.25} = 1.6 > 1$$

Thus, $f(x) = \frac{2}{1+x^2}$ lies above $g(x) = 1$.

On $[1, 3]$, take $\bar{c} = 1.5 \in (1, 3)$

$$f(1.5) = \frac{2}{1+1.25} < 1$$

Thus, $f(x) = \frac{2}{1+x^2}$ lies below $g(x) = 1$ on $[1, 3]$

$$\text{Area} = \int_0^1 (f(x) - g(x)) dx + \int_1^3 (g(x) - f(x)) dx =$$

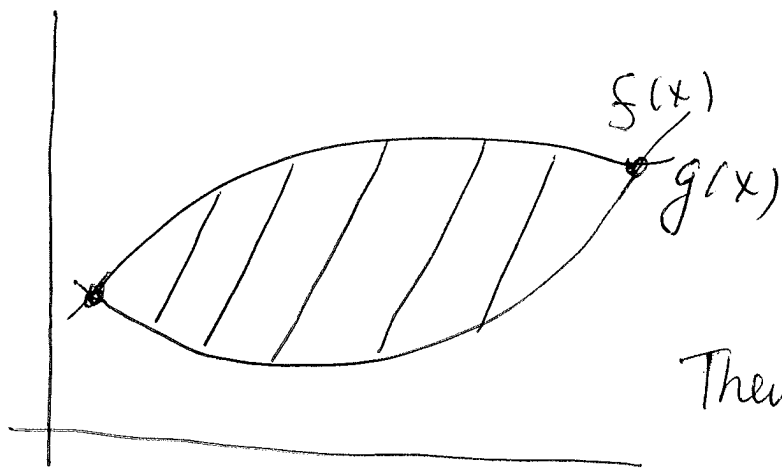
$$= \int_0^1 \left(\frac{2}{1+x^2} - 1 \right) dx + \int_1^3 \left(1 - \frac{2}{1+x^2} \right) dx =$$

$$= 2 \arctan x \Big|_0^1 - x \Big|_0^1 + x \Big|_1^3 - 2 \arctan x \Big|_1^3 =$$

$$= 2 \arctan 1 - 2 \arctan 0 - \underbrace{1 + 0 + 3 - 1} - 2 \arctan 3 +$$
$$+ 2 \arctan 1 = 2 \cdot \frac{\pi}{4} - 2 \cdot 0 + 1 - 2 \arctan 3 + 2 \cdot \frac{\pi}{4} =$$

$$= \pi + 1 - 2 \arctan 3 \approx 1.644$$

The last case:



Need to find the points of intersection (a , and b) (Usually they are not given).

$$\text{Then Area} = \int_a^b |f(x) - g(x)| dx$$

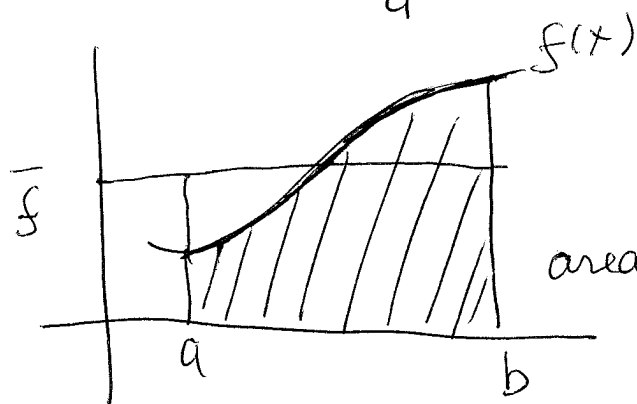
Integrals and Averages

The average value of a function f (denote by \bar{f}) on $[a, b]$ is

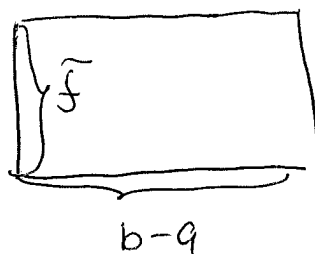
$$\bar{f} = \frac{\int_a^b f(x) dx}{b-a}$$

If $f \geq 0$, then the area under \bar{f} = the area under f

$$\bar{f} \cdot (b-a) = \int_a^b f(x) dx$$



area of



= area of

